Multistability in dynamical systems*

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Abstract

In neuroscience, optics and condensed matter there is ample physical evidence for multistable dynamical systems, that is, systems with a large number of attractors. The known mathematical mechanisms that lead to multiple attractors are homoclinic tangencies and stabilization, by small perturbations or by coupling, of systems possessing a large number of unstable invariant sets. A short review of the existent results is presented, as well as two new results concerning the existence of a large number of stable periodic orbits in a perturbed marginally stable dissipative map and an infinite number of such orbits in two coupled quadratic maps working on the Feigenbaum accumulation point.

1 Multistability. Motivations.

That are three basic motivations to study multistable systems, that is systems that possess a large number of coexisting attractors for a fixed set of parameters. First there is ample evidence for such phenomena in the natural sciences, with examples coming, among others, from neurosciences and neural dynamics [1] [2] [3] [4] [5] [6] [7] , optics [8] [9], chemistry [10] [11] [12], condensed matter [13] and geophysics [14].

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The second motivation to study multistable dynamical systems is the mathematical challenge of identifying the universal mechanisms that lead to multistability and to prove rigorously under what circumstances the phenomenon ,may occur.

The third and last motivation arises from the field of control and technological design. After the pioneering work of Ott, Grebogi and Yorke [15], the field of control of chaos became a whole industry ([16] and references therein). Control of chaos deals with the local control of unstable orbits, either to achieve their stabilization or, alternatively, to target a dynamical system to some desired final state. Reliable stabilization of unstable periodic orbit requires either the knowledge of a good model of the system or an accurate local reconstruction of the dynamics. This is feasible in some low-dimensional systems, but it seems rather problematic for high-dimensional ones. The typical situation in control of chaos, is that of a strange attractor with an infinite number of embedded periodic orbits, all of them unstable. If, instead of an infinite number of unstable periodic orbits, one has, for example, an infinite number of sinks, the controlling situation would seem more promising. The sinks would of course have very small basins of attraction. Nevertheless, the control need not be so accurate, because it suffices to keep the system inside the desired basin of attraction. This, in principle, makes for a more robust control.

In this paper I will concentrate mostly on the rigorous mathematical results that concern multistable dynamical systems. Sects. 2 and 3 contain brief reviews of the Newhouse phenomenon and of the creation of a large number of periodic orbits by dissipative perturbations of conservative systems. Finally, in Sects. 4 and 5, some new results are presented which, in Sect 3, use the techniques of deformation stability and, in Sect. 5, prove the existence of an infinite number of sinks for two coupled quadratic maps.

2 Diffeomorphisms with homoclinic tangencies.The Newhouse phenomenon and beyond

Contrary to earlier conjectures that generic systems might have only finitely many attractors, Newhouse [17] [18] [19] proved that a class of diffeomor-

phisms in a two-dimensional manifold has infinitely many attracting periodic orbits (sinks), a result that was later extended to higher dimensions[20]. Concretely, for two-dimensional manifolds the result is:

Theorem (Newhouse, Robinson[21]) Let f_μ be a C^3 map in a 2-dimensional manifold with C^1 dependence on μ and $\left|\det\left(T_a f^n_{\mu_0}\right)\right| < 1$ and let the non-degenerate homoclinic tangency be crossed at non-zero speed at $\mu = \mu_0$. Then for $\forall \varepsilon > 0$, $\exists (\mu_1, \mu_2) \subset (\mu_0, \mu_0 + \varepsilon)$ and a residual subset $J \subset (\mu_1, \mu_2)$ such that for $\mu \in J$, f_μ has infinitely many sinks.

Models of such diffeomorphisms were constructed by Gambaudo and Tresser[22] and Wang proved that the Newhouse set has positive Hausdorff measure[23].

After these results, intense research followed on the unfolding of homoclinic tangencies and an essential question was whether, in addition to infinitely many sinks, there would also be infinitely many strange attractors near the homoclinic tangencies. The question was positively answered by Colli[24]. The main result is:

Theorem (Colli) Let $f_0 \in Diff^\infty(M)$ be such that f_0 has a homoclinic tangency between the stable and unstable manifolds of a dissipative hyperbolic saddle p_0. Then, there is an open set $\Omega \subset Diff^\infty(M)$ such that

(a) $f_0 \in \overline{\Omega}$

(b) there is a dense subset $D \subset \Omega$ such that for all $f \in D$, f exhibits infinitely many coexisting Hénon-like strange attractors.

Having established the existence of infinitely many sinks and infinitely many strange attractors near homoclinic tangencies, a question of practical importance is the stability of the phenomenon under small random perturbations of the deterministic dynamics. It turns out that the answer to this question is negative. Therefore under small random perturbations only finitely many physical measures will remain.

Theorem (Araújo[25]) Let $f : M \to M$ be a diffeomorphism of class $C^r, r > 1$, of a compact connected boundaryless manifold M of finite dimension. If $f = f_a$ is a member of a parametric family under parametric noise of level $\varepsilon > 0$, that satisfies the hypothesis:

There are $K \in N$ and $\xi_0 > 0$ such that, for all $k \geq K$ and $x \in M$

(A) $f^k(x, \Delta) \supset B^k(x), \xi_0)$;

(B) $f^k(x, \nu^\infty) << m$;

then there is a finite number of probability measures $\mu_1, \cdots \mu_l$ in M with

the properties

1. $\mu_1, \cdots \mu_l$ are physical absolutely continuous probability measures;

2. supp$\mu_i \cap$suppμ_j for all $1 \leq i < j \leq l$;

3. for all $x \in M$ there are open sets $V_1 = V_1(x), \cdots, V_l = V_l(x) \subset \Delta$ such that

(a) $V_i \cap V_j = \emptyset$, $1 \leq i < j \leq l$;

(b) $\nu^\infty \left(\Delta \backslash \left(V_1 \cup \cdots \cup V_l\right)\right) = 0$;

(c) for all $1 \leq i \leq l$ and $\nu^\infty -$ a.e. $t \in V_i$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi\left(f^j(x,t)\right) = \int \phi d\mu$$

for every $\phi \in C\left(M, R\right)$. Moreover the sets $V_1(x), \cdots, V_l(x)$ depend continuously on $x \in M$ with respect to the distance $d_\nu\left(A, B\right) = \nu^\infty\left(A \triangle B\right)$ between $\nu^\infty -$ mod0 subsets of Δ.

3 Small dissipative perturbations of conservative systems

Conservative systems have a large number of coexisting invariant sets, namely periodic orbits, invariant tori and cantori. By adding a small amount of dissipation to a conservative system one finds that some of the invariant sets become attractors. Of course, not all invariant sets of the conservative system will survive when the dissipation is added. However, for sufficiently small dissipation many attractors (mainly periodic orbits) have been observed in typical systems. Poon, Grebogi, Feudel, Hunt and Yorke [26] [27] [28] have extensively studied these effects in the single and double rotor, the Hénon map and the optical cavity map. They find a large number of attractors for a small amount of dissipation, in particular in the double rotor map. The large number of coexisting stable periodic orbits has a complex interwoven basin of attraction structure, with the basin boundaries permeating most of the state space. The chaotic component of the dynamics is in the chaotic saddles embedded in the basin boundary. The systems are also found to be highly sensitive to small amounts of noise. The authors have argued that the two attributes, namely, accessibility to many different states and high sensitivity, are an asset in the sense that they are suitable for an easy control

of the complex system. The transition between different stable states, poses however delicate problems in view of the large chaotic transients in the basin boundary. The problem of migration between attractors and their stability in multiple-attractor systems has also been studied by other authors[29] [30].

All this work is very interesting, however most of results are based on numerical evidence. It would be desirable to have some control of the effects by rigorous mathematical methods. The techniques of deformation stability, to be discussed in the next section, might provide such a tool, in some cases at least.

4 Deformation stability

The basic idea is that, when dissipation is added to a conservative system, it is only a part of the total phase space that is related to the invariant sets of the dissipative system. The question therefore is to find the subsets in the conservative phase space that correspond to the stable sets in the dissipative system. In a sense the situation is similar to KAM theory and to the role played by the constants of motion in this theory. In KAM theory, the first integrals I_0 defined everywhere in phase space for the integrable system, are deformed into a set of constants of motion I_ε which are defined only over a subset of sufficiently irrational tori. By analogy, a similar deformation stability may exist for dissipative perturbations of conservative systems, the stable domain of the constants of motion being, for example, the closure of a family of periodic orbits. In the papers [31] [32] [33] [34] where these ideas were developed, we were concerned with permanence under deformation of invariant sets, not necessarily attracting invariant sets. Most results may however be adapted to the search for attractors. As an example a result will be proved which gives a rigorous criterium for the existence of stable periodic orbits in a class of maps.

Let an $\varepsilon-$family of maps be

$$\begin{aligned} x^{'} &= bx + y + f(x,y,\varepsilon) \\ y^{'} &= y + g(x,y,\varepsilon) \end{aligned}$$

with $f(x,y,0) = g(x,y,0) = 0$ For $\varepsilon = 0$ the map has marginally stable periodic orbits of all periods. Under perturbation some of the orbits become stable ones.

Theorem If f and g are jointly C^2 in (x, y, ε) with $f(x, y, 0) = g(x, y, 0) = 0$, there is an $\overline{\varepsilon}$ such that for $|\varepsilon| < |\overline{\varepsilon}|$ an interior orbit of period p of the unperturbed map becomes a stable orbit of the perturbed map if and only if:

(1) $\sum\limits_{n=0}^{p-1} \partial_\varepsilon g(x_n^{(0)}, y^{(0)}, 0) \mid_{\varepsilon=0} = 0$

(2)

$$\varepsilon \partial_\varepsilon \sum_{n=0}^{p-1} \left\{ \partial_x g(x_n^{(0)}, y^{(0)}, \varepsilon) + (1-b) \partial_y g(x_n^{(0)}, y^{(0)}, \varepsilon) \right\} \mid_{\varepsilon=0} < 0$$

Proof:

Iterating the map p times the orbit condition is:

$$\begin{array}{lcl} (b^p - 1)x + y\frac{1-b^p}{1-b} + u_p(x, y, \varepsilon) - \beta_p(b) & = & 0 \\ v_p(x, y, \varepsilon) = 0 & & \end{array} \tag{1}$$

with

$$\begin{array}{rcl} u_p(x, y, \varepsilon) & = & \sum\limits_{n=0}^{p-1} b^{p-k-1} f(x_n, y_n, \varepsilon) \\ & & +(1-b)^{-1} \sum\limits_{n=0}^{p-2} (1 - b^{p-k-1}) g(x_n, y_n, \varepsilon) \\ v_p(x, y, \varepsilon) & = & \sum\limits_{n=0}^{p-1} g(x_n, y_n, \varepsilon) \end{array} \tag{2}$$

and $\beta_p(b)$ is a polynomial in b.

$b < 1$ and the C^2 condition imply, by the implicit function theorem, the existence of a solution $x = h(y, \varepsilon)$ of the first equation in (1). The second equation becomes

$$v_p(h(y, \varepsilon), y, \varepsilon) = \varepsilon \overline{v_p}(x, y, \varepsilon) = 0 \tag{3}$$

$\overline{v_p}$ being a C^1 function. Condition (1) in the theorem follows from Eq.(3) for $\varepsilon = 0$ and the non-vanishing of the expression in the condition is required by the application of the implicit function theorem to (3).

The eigenvalues of the p-iterated unperturbed map are 1 and b^p. The sign in condition (2) is required to perturb the first eigenvalue to a smaller value. $\square$

5 Coupled quadratic maps

A system as simple as one composed of two coupled quadratic maps, may have an infinite number of stable periodic orbits. Let the system be

$$\begin{array}{c} x_1(t+1) = 1 - \mu_* \left((1-c)x_1(t) + cx_2(t)\right)^2 \\ x_2(t+1) = 1 - \mu_* \left(cx_1(t) + (1-c)x_2(t)\right)^2 \end{array} \tag{4}$$

with $x \in [-1, 1]$, and $\mu_* = 1.401155...$, which is the parameter value of the period doubling accumulation point.

Theorem [35] For sufficiently small c there is an N such that the system (4) has stable periodic orbits of all periods 2^n for $n > N$.

Proof:

Two essential features in the proof are the permanence of the unstabilized orbits in a flip bifurcation and the contraction effect introduced by the convex coupling. Only a sketch of the proof will be presented. For more details refer to Ref.[35].

The bifurcations leading to the Feigenbaum accumulation point at μ_* are flip bifurcations. This means that, after each bifurcation, the orbit that looses stability remains as an unstable periodic orbit. Therefore, (for $c = 0$) at $\mu = \mu_*$ the system (4) has an infinite number of unstable periodic orbits of all periods $p = 2^n$.

The proof has two basic steps. First one proves that, for sufficiently small $c \neq 0$, these periodic orbits still exist in the system (4). Second, that for any such c, there is an N such that there is at least one stable orbit for all periods $p = 2^n$ with $n > N$. For both steps an important role is played by the instability factor, given by $\left(f^{(p)}\right)'(x_p)$ at the fixed points x_p of the $p-$iterated map. Using the properties of the Feigenbaum - Cvitanovic functional equation one finds that the instability factor $\left(f^{(p)}\right)'(x^*)$ converges to a fixed non-zero uniformly bounded value for all orbits. One now proceeds to the proof of the theorem.

First step: Permanence of the periodic orbits for small c

Let $x_p^* \in [-1, 1] \times [-1, 1]$ be, for example, the coordinate of the $p-$periodic orbit closest to zero. The sequence $\{x^*\} = \left\{x_p^* : p = 2, 4, 8, ...\right\}$ is an element of a ℓ_∞ Banach space (sup norm). The collection of fixed point equations

$$f_{\mu^*}^{(p)}(x_p^*, c) - x_p^* = 0$$

defines a $C^\infty-$mapping $F(x^*, c)$ from $\ell_\infty \times R \rightarrow \ell_\infty$. Because $\left(f^{(p)}\right)^{'}(x^*)$ at $c = 0$ is negative and bounded for all p, the derivative $D_1 F$ of the mapping in the first argument is invertible. Therefore, by the implicit function theorem for Banach spaces, there is a c^* such that for $c < c^*$ the function $x^*(c) : R \rightarrow \ell_\infty$ is defined, that is, there are $p-$periodic orbits for all periods $p = 2^n$.

For the uncoupled case ($c = 0$) the instability factor $\left(f^{(p)}\right)^{'}(x^*)$ for each mapping is the product $(-2\mu_*)^p \prod_{k=1}^p x(k)$ over the orbit coordinates. For $c < c^*$, the orbit structure being preserved, their projections on the axis are continuous deformations of the $c = 0$ case which will preserve the geometric relations of the Feigenbaum accumulation point. Hence the same products for the projected coordinates suffer changes of order $a(c)p\lambda^p$ and remain bounded.

Second step: Stabilization of at least one orbit for all periods $p = 2^n$ with $n > N(c)$

The stability of the periodic orbits is controlled by the eigenvalues of the Jacobian $J_p = \frac{Df^{(p)}_{\mu_*,c}}{Dx}$ in the fixed point of $f^{(p)}_{\mu_*,c}$. The map (4) is a composition of two maps $f_1 \circ f_2$

$$f_1 : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 - \mu_* x_1^2 \\ 1 - \mu_* x_2^2 \end{pmatrix}$$

$$f_2 : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \begin{pmatrix} (1-c)x_1 + cx_2 \\ cx_1 + (1-c)x_2 \end{pmatrix}$$

and by the chain rule the Jacobian has determinant

$$\det J_p = (1-2c)^p (-2\mu_*)^{2p} \prod_{k=1}^{p} x_1(k) x_2(k) \tag{5}$$

Because of the permanence of the periodic orbits, for small c, in the neighborhood of the original coordinates (those for $c = 0$), the product of the last two factors in (5) is uniformly bounded for all p. Then for all sufficiently large p, $|\det J_p| < 1$. The question is how this overall contraction is distributed among the two eigenvalues of J_p.

To discuss the nature of the eigenvalues we may use a first order approximation in c. For a periodic orbit of period $p = 2^n$ we define

$$X_{l,q}^{(i)} = \left\{ \begin{array}{ll} (-2\mu_*)^{q-l+1} \prod_{k=l}^{q} x_i(k) & \text{if } q \geq l \\ 1 & \text{if } l > q \end{array} \right\} \tag{6}$$

For small c consider the linear approximation to the Jacobian J_p

$$\begin{pmatrix} (1-pc)X_{1,p}^{(1)} & c\sum_{k=1}^{p} X_{1,k}^{(1)} X_{k+1,p}^{(2)} \\ c\sum_{k=1}^{p} X_{1,k}^{(2)} X_{k+1,p}^{(1)} & (1-pc)X_{1,p}^{(2)} \end{pmatrix}$$

The eigenvalues are

$$\begin{aligned} \lambda_{\pm} &= \tfrac{1}{2}(1-pc)\left(X_{1,p}^{(1)}+X_{1,p}^{(2)}\right) \\ &\pm\tfrac{1}{2}\sqrt{(1-pc)^2\left(X_{1,p}^{(1)}-X_{1,p}^{(2)}\right)^2+4c^2\sum_{k=1}^{p} X_{1,k}^{(1)} X_{k+1,p}^{(2)} \sum_{k'=1}^{p} X_{1,k'}^{(2)} X_{k'+1,p}^{(1)}} \end{aligned} \tag{7}$$

If the periodic orbit runs with the two coordinates x_1 and x_2 synchronized then

$$\begin{aligned} \lambda_+ &= X_{1,p} \\ \lambda_- &= (1-2pc)\,X_{1,p} \end{aligned}$$

and the orbit being unstable for $c=0$ it remains unstable for $c\neq 0$. However if the two coordinates are out of phase by $\frac{p}{2}$ steps the radical in Eq.(7) is

$$\sqrt{\prod_{k=1}^{p} x(k)}\left|\sum_{i=1}^{p} x(i)x(i+1)\cdots x(i+\frac{p}{2}-1)\right|$$

The existence of a superstable orbit for all periods $p=2^n$ implies that at $\mu=\mu_*$ the product $\prod_{k=1}^{p} x(k)$ has an odd number of negative-valued coordinates. Therefore the two eigenvalues are complex conjugate and, for small c, the contraction implicit in (5) is equally distributed by the two eigenvalues. Therefore for sufficiently large N all orbits of this type with $p>2^N$ become stable periodic orbits. □

The attracting periodic orbits of the coupled system being associated to the unstable periodic orbits of the Feigenbaum cascade, the basins of attraction will be controlled by the neighborhoods of these orbits, in each coordinate. Therefore a checkerboard-type structure is expected for the basins of attraction.

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